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# Thermomechanical postbuckling of composite laminated cylindrical shells with local geometric imperfections

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**Abstract**

The effect of local geometric imperfections on the buckling and postbuckling of composite laminated cylindrical shells subjected to combined axial compression and uniform temperature loading was investigated. The two cases of compressive postbuckling of initially heated shells and of thermal postbuckling of initially compressed shells are considered. The formulations are based on a boundary layer theory of shell buckling, which includes the effects of the nonlinear prebuckling deformation, the nonlinear large deflection in the postbuckling range and the initial geometric imperfection of the shell. The analysis uses a singular perturbation technique to determine buckling loads and postbuckling equilibrium paths. Numerical examples are presented that relate to the performances of cross-ply laminated cylindrical shells with or without initial local imperfections, from which results for isotropic cylindrical shells follow as a limiting case. Typical results are presented in dimensionless graphical form for different parameters and loading conditions.


*Key words:* Structural stability; Thermomechanical postbuckling; Composite laminated cylindrical shell; A boundary layer theory of shell buckling; Singular perturbation technique

## Nomenclature

| | |
|---|---|
| $E_{11}$, $E_{22}$ | elastic moduli for single ply |
| $\bar{F}$, $F$ | stress function and its dimensionless form |
| $G_{12}$ | shear modulus for single ply |
| $L$ | length of shell |
| $R$ | radius of cylindrical shell |
| $t$ | thickness of shell |
| $\bar{W}$, $W$ | deflection of shell and its dimensionless form |
| $\bar{W}^*$, $W^*$ | geometrical imperfection of shell and its dimensionless form |
| $Z$ | geometric parameter of shell, $=L^2/Rt$ |
| $\alpha_{11}$, $\alpha_{22}$ | thermal expansion coefficients for single ply |
| $\Delta_x$, $\delta_x$, $\delta_P^*$ | average end-shortening and its two alternative dimensionless forms |


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| | |
|---|---|
| $\varepsilon$ | small perturbation parameter |
| $\lambda^*$ | imperfection sensitivity parameter |
| $\lambda_p$, $\lambda_p^*$ | dimensionless forms of axial compressive load |
| $\lambda_T$, $\lambda_T^*$ | dimensionless forms of thermal load |
| $\mu$ | imperfection parameter |
| $\nu_{12}$, $\nu_{21}$ | Poisson's ratios |

## 1. Introduction

The postbuckling response of multilayered composite cylindrical shells subjected to combined axial and thermal loads is of current interest to engineers engaged in nuclear, petrochemical and aerospace engineering practice. These cylindrical shells may have significant and unavoidable initial geometric imperfections. Although imperfection distributions are likely to be random in nature, it is often observed that local dimples or modal imperfections are presented in the shell structure. Therefore, there is a need to understand the postbuckling behavior of imperfect composite laminated cylindrical shells under combined axial compression and thermal loading.

Many initial postbuckling or fully nonlinear postbuckling studies have been made for isotropic and multilayered composite cylindrical shells with modal imperfections, see, for example, Arbocz and Babcock (1969), Sheinman et al. (1983) and Shulga et al. (1992), whereas relatively few have been made for cylindrical shells with local geometric imperfections. The effect of a cosine local imperfection on the buckling of cylindrical shells under axial compression was studied by Hutchinson et al. (1971). Amazigo and Budiansky (1972) gave an imperfection sensitivity analysis of axially compressed cylindrical shells with localized axisymmetric imperfections using Koiter's initial postbuckling theory (see Koiter, 1945). The effect of large diamond shaped dimples on the buckling of cylindrical shells under axial compression was investigated experimentally by Krishnakumar and Foster (1991). For pressurized cylindrical shells the effect of dimple imperfections was performed by Amazigo and Fraser (1971) and Abdelmoula et al. (1992). All of the investigations mentioned above are concerned with isotropic cylindrical shells. To the best of the author's knowledge, no papers deal with the postbuckling of composite laminated cylindrical shells with local geometric imperfections subjected to combined axial and thermal loads.

It has been shown by Bushnell and Smith (1971) that in shell thermal buckling as well as in shell compressive buckling there is a boundary layer phenomenon where prebuckling and buckling displacements vary rapidly. Based on a boundary layer theory of shell buckling suggested by Shen and Chen (1988, 1990), which includes the effects of nonlinear prebuckling deformations, large deflections in the postbuckling range and initial geometric imperfections of the shell, a postbuckling analysis of perfect and imperfect, stiffened and unstiffened, isotropic and composite laminated cylindrical shells under various loading cases has been presented by Shen and Chen (1991), Shen et al. (1993) and Shen (1997a–c). The present study extends the previous work to the case of composite laminated cylindrical shells with local geometric imperfections subjected to combined axial compression and uniform temperature loading. The material properties are assumed to be independent of temperature. The nonlinear prebuckling deformations and the initial local geometric imperfections of the shell are both taken into account. The analysis uses a singular perturbation technique to determine the required buckling loads and postbuckling equilibrium paths.

## 2. Analytical formulation

Consider a thin cylindrical shell with mean radius $R$, length $L$ and thickness $t$, which consists of $N$ plies, subjected to two loads combined out of axial compression $P_x$ and uniform temperature rise $T_0$. Let $\bar{U}$, $\bar{V}$ and $\bar{W}$ be the displacements parallel to a right-hand set of axes $(X, Y, Z)$, where $X$, $Y$ and $Z$ are the axial, circumferential and radial (positive inward) coordinates on shell middle surface. Denoting the initial deflection by $\bar{W}^*(X, Y)$, let $\bar{W}(X, Y)$ be the additional deflection and $\bar{F}(X, Y)$ be the stress function for the stress resultants, and denoting differentiation by a comma, so that $N_x = \bar{F}_{,yy}$, $N_y = \bar{F}_{,xx}$, $N_{xy} = -\bar{F}_{,xy}$.

Attention is confined to the case of cross-ply laminated cylindrical shells, from which solutions for isotropic or orthotropic cylindrical shells follow as a limiting case.

From classical laminated shell theory (i.e. transverse shear deformation effects are neglected) and including thermal effects, leads to the governing differential equations (see Shen, 1997c). They are

$$L_1(\bar{W})+L_3(\bar{F})-L_4(N^T)-L_6(M^T)-\frac{1}{R}\bar{F}_{,xx} = L(\bar{W}+\bar{W}^*,\bar{F}) \tag{1}$$

$$L_2(\bar{F})-L_3(\bar{W})-L_5(N^T)+\frac{1}{R}\bar{W}_{,xx} = -\frac{1}{2}L(\bar{W}+2\bar{W}^*,\bar{W}) \tag{2}$$

where

$$L_1() = D^*_{11}\frac{\partial^4}{\partial X^4}+2(D^*_{12}+2D^*_{66})\frac{\partial^4}{\partial X^2\,\partial Y^2}+D^*_{22}\frac{\partial^4}{\partial Y^4}$$

$$L_2() = A^*_{22}\frac{\partial^4}{\partial X^4}+(2A^*_{12}+A^*_{66})\frac{\partial^4}{\partial X^2\,\partial Y^2}+A^*_{11}\frac{\partial^4}{\partial Y^4}$$

$$L_3() = B^*_{21}\frac{\partial^4}{\partial X^4}+(B^*_{11}+B^*_{22}-2B^*_{66})\frac{\partial^4}{\partial X^2\,\partial Y^2}+B^*_{12}\frac{\partial^4}{\partial Y^4}$$

$$L() = \frac{\partial^2}{\partial X^2}\frac{\partial^2}{\partial Y^2}-2\frac{\partial^2}{\partial X\,\partial Y}\frac{\partial^2}{\partial X\,\partial Y}+\frac{\partial^2}{\partial Y^2}\frac{\partial^2}{\partial X^2}$$

$$L_4(N^T) = \left(B^*_{11}\frac{\partial^2}{\partial X^2}+B^*_{12}\frac{\partial^2}{\partial Y^2}\right)N^T_x+2B^*_{66}\frac{\partial^2}{\partial X\,\partial Y}(N^T_{xy})+\left(B^*_{21}\frac{\partial^2}{\partial X^2}+B^*_{22}\frac{\partial^2}{\partial Y^2}\right)N^T_y$$

$$L_5(N^T) = \left(A^*_{12}\frac{\partial^2}{\partial X^2}+A^*_{11}\frac{\partial^2}{\partial Y^2}\right)N^T_x-A^*_{66}\frac{\partial^2}{\partial X\,\partial Y}(N^T_{xy})+\left(A^*_{22}\frac{\partial^2}{\partial X^2}+A^*_{12}\frac{\partial^2}{\partial Y^2}\right)N^T_y$$

$$L_6(M^T) = \frac{\partial^2}{\partial X^2}(M^T_x)+2\frac{\partial^2}{\partial X\,\partial Y}(M^T_{xy})+\frac{\partial^2}{\partial Y^2}(M^T_y) \tag{3}$$

in these equations $[A^*_{ij}]$, $[B^*_{ij}]$ and $[D^*_{ij}]$ $(i,j = 1,2,6)$ are reduced stiffness matrices defined as

$\mathbf{A}^* = \mathbf{A}^{-1}$, $\mathbf{B}^* = -\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{D}^* = \mathbf{D}-\mathbf{B}\mathbf{A}^{-1}\mathbf{B}$, where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ are membrane, coupling and flexural rigidities of the shell and details of which can be found in Appendix A.

The thermal forces and moments caused by the temperature rise $T_0$ are defined by

$$\begin{bmatrix} N_x^T, & M_x^T \\ N_y^T, & M_y^T \\ N_{xy}^T, & M_{xy}^T \end{bmatrix} = \sum_{k=1} \int_{t_{k-1}}^{t_k} (1, Z) \begin{bmatrix} A_x \\ A_y \\ A_{xy} \end{bmatrix}_k T_0 \, \mathrm{d}Z \tag{4}$$

It is noted that the operator $L_3()$ in eqns (1) and (2) includes the coupling between transverse bending and in-plane stretching and the operator $L_4()$ in eqn (1) includes the thermal coupling. From eqns (3) and (4) it is clear that $N_{xy}^T$ and $M^T$ are all zero, and $N_x^T$ and $N_y^T$ are both constants, so that $L_4(N^T) = L_5(N^T) = L_6(M^T) = 0$.

The average end-shortening is defined as

$$\begin{aligned} \frac{\Delta_x}{L} &= -\frac{1}{2\pi RL}\int_0^{2\pi R}\int_{-L/2}^{+L/2} \frac{\partial \bar{U}}{\partial X}\mathrm{d}X\,\mathrm{d}Y \\ &= -\frac{1}{2\pi RL}\int_0^{2\pi R}\int_{-L/2}^{+L/2}\left[\left(A_{11}^*\frac{\partial^2 \bar{F}}{\partial Y^2}+A_{12}^*\frac{\partial^2 \bar{F}}{\partial X^2}\right)-\left(B_{11}^*\frac{\partial^2 \bar{W}}{\partial X^2}+B_{12}^*\frac{\partial^2 \bar{W}}{\partial Y^2}\right)-\frac{1}{2}\left(\frac{\partial \bar{W}}{\partial X}\right)^2\right. \\ &\qquad \left. -\frac{\partial \bar{W}}{\partial X}\frac{\partial \bar{W}^*}{\partial X}-(A_{11}^* N_x^T + A_{12}^* N_y^T)\right]\mathrm{d}X\,\mathrm{d}Y \end{aligned} \tag{5}$$

and we have the closed (or periodicity) condition

$$\int_0^{2\pi R}\frac{\partial \bar{V}}{\partial Y}\mathrm{d}Y = 0 \tag{6a}$$

or

$$\begin{aligned} &\int_0^{2\pi R}\left[\left(A_{22}^*\frac{\partial^2 \bar{F}}{\partial X^2}+A_{12}^*\frac{\partial^2 \bar{F}}{\partial Y^2}\right)-\left(B_{21}^*\frac{\partial^2 \bar{W}}{\partial X^2}+B_{22}^*\frac{\partial^2 \bar{W}}{\partial Y^2}\right)+\frac{\bar{W}}{R}-\frac{1}{2}\left(\frac{\partial \bar{W}}{\partial Y}\right)^2-\frac{\partial \bar{W}}{\partial Y}\frac{\partial \bar{W}^*}{\partial Y}\right. \\ &\qquad\qquad \left.-(A_{12}^* N_x^T + A_{22}^* N_y^T)\right]\mathrm{d}Y = 0 \end{aligned} \tag{6b}$$

Two loading cases are considered. In the first case, a uniform temperature rise is complemented by increasing mechanical compressive edge loading. In the second case, mechanical compressive loading is kept at a constant prebuckling level and the ends of the shell are assumed to be restrained against expansion longitudinally while the uniform temperature is increased steadily. As a result, the boundary conditions are

$$X = \pm L/2;$$

$$\bar{W} = 0, \quad \bar{M}_x = -B_{11}^*\frac{\partial^2 \bar{F}}{\partial Y^2} - B_{21}^*\frac{\partial^2 \bar{F}}{\partial X^2} - D_{11}^*\frac{\partial^2 \bar{W}}{\partial X^2} - D_{12}^*\frac{\partial^2 \bar{W}}{\partial Y^2} + M_x^T = 0$$

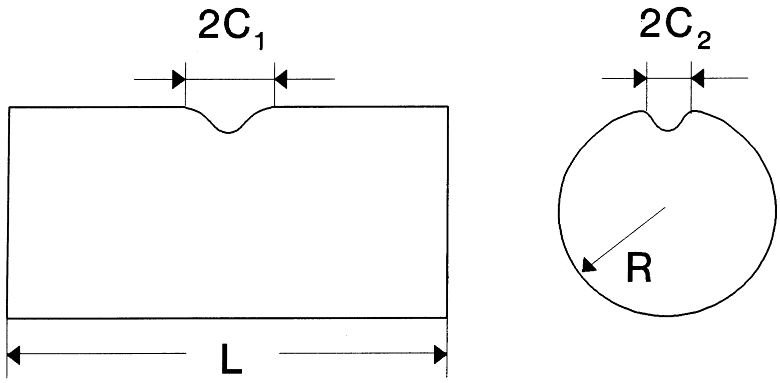



Fig. 1. A cylindrical shell with a local geometric imperfection.

(simply supported) (7a)

$$\bar{W} = \bar{W}_{,x} = 0 \quad \text{(clamped)} \tag{7b}$$

$$\int_0^{2\pi R} N_x \, dY + P_x = 0 \quad \text{(for compressive buckling problem)} \tag{7c}$$

$$\bar{U} = 0 \quad \text{(for thermal buckling problem)} \tag{7d}$$

A local asymmetric imperfection is to be assumed as (Fig. 1)

$$\bar{W}^*(X, Y) = A_m \exp\left(-\left|\frac{X}{C_1}\right| - \left|\frac{Y}{C_2}\right|\right) \tag{8}$$

where $A_m$ is a small parameter characterizing the amplitude of the initial imperfection and $C_1$ and $C_2$ characterize the half-width of the region of the dimple. Thus local means here that the initial deflection decay exponentially in both $X$- and $Y$-directions.

Equations (1)–(8) are the governing equations describing the required large deflection post-buckling response of the shell.

## 3. Analytical method and asymptotic solutions

Let the thermal expansion coefficients for each ply be

$$\alpha_{11} = a_{11}\alpha_0, \quad \alpha_{22} = a_{22}\alpha_0 \tag{9}$$

where $\alpha_0$ is an arbitrary reference value, and let

$$\begin{bmatrix} A_x^T \\ A_y^T \end{bmatrix} = -\sum_{k=1} \int_{t_{k-1}}^{t_k} \begin{bmatrix} A_x \\ A_y \end{bmatrix}_k dZ \tag{10}$$

Introducing the dimensionless quantities (in which the alternative forms $\lambda_P^*$, $\delta_P^*$ and $\lambda_T^*$ are not needed until the numerical examples are considered)

$$x = \pi X/L, \quad y = Y/R, \quad \beta = L/\pi R, \quad \bar{Z} = L^2/Rt, \quad \varepsilon = (\pi^2 R/L^2)[D_{11}^* D_{22}^* A_{11}^* A_{22}^*]^{1/4},$$

$$(W, W^*) = \varepsilon(\bar{W}, \bar{W}^*)/[D_{11}^* D_{22}^* A_{11}^* A_{22}^*]^{1/4}, \quad F = \varepsilon^2 \bar{F}/[D_{11}^* D_{22}^*]^{1/2},$$

$$\gamma_{12} = (D_{12}^* + 2D_{66}^*)/D_{11}^*, \quad \gamma_{14} = [D_{22}^*/D_{11}^*]^{1/2},$$

$$\gamma_{22} = (A_{12}^* + A_{66}^*/2)/A_{22}^*, \quad \gamma_{24} = [A_{11}^*/A_{22}^*]^{1/2}, \quad \gamma_5 = -A_{12}^*/A_{22}^*,$$

$$(\gamma_{30}, \gamma_{32}, \gamma_{34}, \gamma_{311}, \gamma_{322}) = (B_{21}^*, B_{11}^* + B_{22}^* - 2B_{66}^*, B_{12}^*, B_{11}^*, B_{22}^*)/[D_{11}^* D_{22}^* A_{11}^* A_{22}^*]^{1/4},$$

$$(\gamma_{T1}, \gamma_{T2}) = (A_x^T, A_y^T)R/\alpha_0[D_{11}^* D_{22}^*/A_{11}^* A_{22}^*]^{1/4}, \quad (\gamma_1, \gamma_2) = (L/\pi C_1, R/C_2),$$

$$M_x = \varepsilon^2 \bar{M}_x L^2/\pi^2 D_{11}^* [D_{11}^* D_{22}^* A_{11}^* A_{22}^*]^{1/4},$$

$$\lambda_P = P_x/4\pi[D_{11}^* D_{22}^*/A_{11}^* A_{22}^*]^{1/4}, \quad \lambda_P^* = P_x[3(1-\nu_{12}\nu_{21})]^{1/2}/2\pi t^2[E_{11}E_{22}]^{1/2},$$

$$\delta_x = (\Delta_x/L)/(2/R)[D_{11}^* D_{22}^* A_{11}^* A_{22}^*]^{1/4}, \quad \delta_P^* = (\Delta_x/L)[3(1-\nu_{12}\nu_{21})]^{1/2}/(t/R),$$

$$(\lambda_T, \lambda_T^*) = (1, 10^3)\alpha_0 T_0 \tag{11}$$

enables the nonlinear eqns (1) and (2) to be written in dimensionless form as

$$\varepsilon^2 L_1(W) + \varepsilon\gamma_{14} L_3(F) - \gamma_{14} F_{,xx} = \gamma_{14}\beta^2 L(W + W^*, F) \tag{12}$$

$$L_2(F) - \varepsilon\gamma_{24} L_3(W) + \gamma_{24} W_{,xx} = -\tfrac{1}{2}\gamma_{24}\beta^2 L(W + 2W^*, W) \tag{13}$$

where

$$L_1(\,) = \frac{\partial^4}{\partial x^4} + 2\gamma_{12}\beta^2 \frac{\partial^4}{\partial x^2\,\partial y^2} + \gamma_{14}^2\beta^4 \frac{\partial^4}{\partial y^4}$$

$$L_2(\,) = \frac{\partial^4}{\partial x^4} + 2\gamma_{22}\beta^2 \frac{\partial^4}{\partial x^2\,\partial y^2} + \gamma_{24}^2\beta^4 \frac{\partial^4}{\partial y^4}$$

$$L_3(\,) = \gamma_{30}\frac{\partial^4}{\partial x^4} + \gamma_{32}\beta^2 \frac{\partial^4}{\partial x^2\,\partial y^2} + \gamma_{34}\beta^4 \frac{\partial^4}{\partial y^4}$$

$$L(\,) = \frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2} - 2\frac{\partial^2}{\partial x\,\partial y}\frac{\partial^2}{\partial x\,\partial y} + \frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial x^2} \tag{14}$$

The unit end-shortening relationship becomes

$$\delta_x = -\frac{1}{4\pi^2\gamma_{24}}\varepsilon^{-1}\int_0^{2\pi}\int_{-\pi/2}^{+\pi/2}\Bigg[\left(\gamma_{24}^2\beta^2\frac{\partial^2 F}{\partial y^2} - \gamma_5\frac{\partial^2 F}{\partial x^2}\right) - \varepsilon\gamma_{24}\left(\gamma_{311}\frac{\partial^2 W}{\partial x^2} + \gamma_{34}\beta^2\frac{\partial^2 W}{\partial y^2}\right)$$
$$- \frac{1}{2}\gamma_{24}\left(\frac{\partial W}{\partial x}\right)^2 - \gamma_{24}\frac{\partial W}{\partial x}\frac{\partial W^*}{\partial x} + (\gamma_{24}^2\gamma_{T1} - \gamma_5\gamma_{T2})\lambda_T\varepsilon\Bigg]\mathrm{d}x\,\mathrm{d}y \tag{15}$$

and the closed condition becomes

$$\int_0^{2\pi}\left[\left(\frac{\partial^2 F}{\partial x^2}-\gamma_5\beta^2\frac{\partial^2 F}{\partial y^2}\right)-\varepsilon\gamma_{24}\left(\gamma_{30}\frac{\partial^2 W}{\partial x^2}+\gamma_{322}\beta^2\frac{\partial^2 W}{\partial y^2}\right)+\gamma_{24}W\right.$$
$$\left.-\frac{1}{2}\gamma_{24}\beta^2\left(\frac{\partial W}{\partial y}\right)^2-\gamma_{24}\beta^2\frac{\partial W}{\partial y}\frac{\partial W^*}{\partial y}+(\gamma_{T2}-\gamma_5\gamma_{T1})\lambda_T\varepsilon\right]\mathrm{d}y=0 \quad (16)$$

Note that eqns (12) and (13) are identical to those of composite laminated cylindrical shells under pure axial compression, but that eqn (11) is augmented by the definition of $\lambda_T$, which is used in eqns (15) and (16).

The boundary conditions become $x = \pm\pi/2$;

$$W = M_x = 0 \quad \text{(simply supported)} \quad (17a)$$

$$W = W_{,x} = 0 \quad \text{(clamped)} \quad (17b)$$

$$\frac{1}{2\pi}\int_0^{2\pi}\beta^2\frac{\partial^2 F}{\partial y^2}\mathrm{d}y+2\lambda_P\varepsilon=0 \quad \text{(for compressive buckling problem)} \quad (17c)$$

$$\delta_x = 0 \quad \text{(for thermal buckling problem)} \quad (17d)$$

For isotropic cylindrical shells, because of eqn (11) we have $\varepsilon = \pi^2/\bar{Z}_B\sqrt{12}$, where $\bar{Z}_B = (L^2/Rt)[1-v^2]^{1/2}$ is Batdorf shell parameter which should be greater than 2.85 in the case of classical buckling analysis (see Batdorf, 1947). $\bar{Z}_B > 2.85$ results $\varepsilon < 1$,then eqns (12) and (13) are the equations of the boundary layer type, from which nonlinear prebuckling deformations, large deflections in the postbuckling range and initial geometric imperfections of the shell can be considered simultaneously.

Applying eqns (12)–(17), the thermomechanical postbuckling behavior of such shells is now determined by a singular perturbation technique suggested in Shen and Chen (1988, 1990).

To construct an asymptotic solution for the composite laminated cylindrical shell, the additional deflection and stress functions in eqns (12) and (13) are assumed as

$$W = w(x,y,\varepsilon)+\tilde{W}(x,\xi,y,\varepsilon)+\hat{W}(x,\zeta,y,\varepsilon)$$
$$F = f(x,y,\varepsilon)+\tilde{F}(x,\xi,y,\varepsilon)+\hat{F}(x,\zeta,y,\varepsilon) \quad (18)$$

where $\varepsilon$ is a small perturbation parameter defined in eqn (11) and $w(x,y,\varepsilon)$, $f(x,y,\varepsilon)$ are called outer solutions or regular solutions of the shell, $\tilde{W}(x,\xi,y,\varepsilon)$, $\tilde{F}(x,\xi,y,\varepsilon)$ and $\hat{W}(x,\zeta,y,\varepsilon)$, $\hat{F}(x,\zeta,y,\varepsilon)$ are boundary layer solutions near the $x = \pm\pi/2$ edges, respectively, and $\xi$ and $\zeta$ are the boundary layer variables, defined as

$$\xi = \frac{\pi/2+x}{\sqrt{\varepsilon}}, \quad \zeta = \frac{\pi/2-x}{\sqrt{\varepsilon}} \quad (19)$$

(This means for isotropic cylindrical shells the width of the boundary layers is of the order $\sqrt{Rt}$.) In eqn (18) the regular and boundary layer solutions are taken in the form of perturbation expansions as

$$w(x,y,\varepsilon) = \sum_{j=1}\varepsilon^j w_j(x,y), \quad f(x,y,\varepsilon) = \sum_{j=0}\varepsilon^j f_j(x,y) \quad (20a)$$

$$\tilde{W}(x,\xi,y,\varepsilon) = \sum_{j=0} \varepsilon^{j+1} \tilde{W}_{j+1}(x,\xi,y), \quad \tilde{F}(x,\xi,y,\varepsilon) = \sum_{j=0} \varepsilon^{j+2} \tilde{F}_{j+2}(x,\xi,y) \tag{20b}$$

$$\hat{W}(x,\zeta,y,\varepsilon) = \sum_{j=0} \varepsilon^{j+1} \hat{W}_{j+1}(x,\zeta,y), \quad \hat{F}(x,\zeta,y,\varepsilon) = \sum_{j=0} \varepsilon^{j+2} \hat{F}_{j+2}(x,\zeta,y) \tag{20c}$$

The initial buckling mode is assumed to have the form

$$w_2(x,y) = A_{11}^{(2)} \cos mx \cos ny \tag{21}$$

The initial local geometric imperfection is represented as a Fourier cosine series, i.e.

$$\begin{aligned} W^*(x,y,\varepsilon) &= \varepsilon^2 a_m \exp(-\gamma_1|x| - \gamma_2|y|) \\ &= \varepsilon^2 \mu A_{11}^{(2)} \left(\frac{a_0}{2} + \sum_{i=1} a_i \cos ix\right)\left(\frac{b_0}{2} + \sum_{j=1} b_j \cos jy\right) \end{aligned} \tag{22a}$$

where

$$a_i = \frac{4}{\pi}\int_0^{\pi/2} \exp(-\gamma_1 x)\cos ix \,\mathrm{d}x, \quad b_j = \frac{2}{\pi}\int_0^{\pi} \exp(-\gamma_2 y)\cos jy \,\mathrm{d}y \tag{22b}$$

and $\mu = a_m/A_{11}^{(2)}$ is the imperfection parameter.

Substituting eqns (18)–(20) into eqns (12) and (13) gives a set of perturbation equations that can be solved step by step.

Then using eqns (21) and (22) to solve these perturbation equations of each order, and matching the regular solutions with the boundary layer solutions at each end of the shell, so that the asymptotic solutions satisfying clamped boundary conditions are constructed as

$$\begin{aligned} W(x,y,\varepsilon) = \varepsilon\Bigg[ & A_{00}^{(1)} - A_{00}^{(1)}\left(\cos\phi\frac{\pi/2+x}{\sqrt{\varepsilon}} + \frac{\alpha}{\phi}\sin\phi\frac{\pi/2+x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi/2+x}{\sqrt{\varepsilon}}\right) \\ & - A_{00}^{(1)}\left(\cos\phi\frac{\pi/2-x}{\sqrt{\varepsilon}} + \frac{\alpha}{\phi}\sin\phi\frac{\pi/2-x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi/2-x}{\sqrt{\varepsilon}}\right)\Bigg] \\ & + \varepsilon^2\Bigg[A_{11}^{(2)}\cos mx\cos ny + A_{20}^{(2)}\cos 2mx + A_{02}^{(2)}\cos 2ny \\ & - (-A_{20}^{(2)} + A_{02}^{(2)}\cos 2ny)\left(\cos\phi\frac{\pi/2+x}{\sqrt{\varepsilon}} + \frac{\alpha}{\phi}\sin\phi\frac{\pi/2+x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi/2+x}{\sqrt{\varepsilon}}\right) \\ & - (-A_{20}^{(2)} + A_{02}^{(2)}\cos 2ny)\left(\cos\phi\frac{\pi/2-x}{\sqrt{\varepsilon}} + \frac{\alpha}{\phi}\sin\phi\frac{\pi/2-x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi/2-x}{\sqrt{\varepsilon}}\right)\Bigg] \\ & + \varepsilon^3[A_{11}^{(3)}\cos mx\cos ny + A_{02}^{(3)}\cos 2ny] + \varepsilon^4[A_{00}^{(4)} + A_{20}^{(4)}\cos 2mx + A_{02}^{(4)}\cos 2ny \\ & + A_{13}^{(4)}\cos mx\cos 3ny + A_{04}^{(4)}\cos 4ny] + O(\varepsilon^5) \end{aligned} \tag{23}$$

$$F(x,y,\varepsilon) = -B_{00}^{(0)}\frac{y^2}{2}+\varepsilon\left[-B_{00}^{(1)}\frac{y^2}{2}\right]+\varepsilon^2\left[-B_{00}^{(2)}\frac{y^2}{2}+B_{11}^{(2)}\cos mx\cos ny\right.$$

$$+A_{00}^{(1)}\left(\gamma_{24}\left(\frac{1}{b}+\gamma_{30}\right)\cos\phi\frac{\pi/2+x}{\sqrt{\varepsilon}}-\gamma_{24}\left(\frac{1}{b}-\gamma_{30}\right)\frac{\alpha}{\phi}\sin\phi\frac{\pi/2+x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi/2+x}{\sqrt{\varepsilon}}\right)$$

$$\left.+A_{00}^{(1)}\left(\gamma_{24}\left(\frac{1}{b}+\gamma_{30}\right)\cos\phi\frac{\pi/2-x}{\sqrt{\varepsilon}}-\gamma_{24}\left(\frac{1}{b}-\gamma_{30}\right)\frac{\alpha}{\phi}\sin\phi\frac{\pi/2-x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi/2-x}{\sqrt{\varepsilon}}\right)\right]$$

$$+\varepsilon^3\left[-B_{00}^{(3)}\frac{y^2}{2}+B_{02}^{(3)}\cos 2ny+(-A_{20}^{(2)}+A_{02}^{(2)}\cos 2ny)\left(\gamma_{24}\left(\frac{1}{b}+\gamma_{30}\right)\cos\phi\frac{\pi/2+x}{\sqrt{\varepsilon}}\right.\right.$$

$$\left.-\gamma_{24}\left(\frac{1}{b}-\gamma_{30}\right)\frac{\alpha}{\phi}\sin\phi\frac{\pi/2+x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi/2+x}{\sqrt{\varepsilon}}\right)$$

$$+(-A_{20}^{(2)}+A_{02}^{(2)}\cos 2ny)\left(\gamma_{24}\left(\frac{1}{b}+\gamma_{30}\right)\cos\phi\frac{\pi/2-x}{\sqrt{\varepsilon}}\right.$$

$$\left.\left.-\gamma_{24}\left(\frac{1}{b}-\gamma_{30}\right)\frac{\alpha}{\phi}\sin\phi\frac{\pi/2-x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi/2-x}{\sqrt{\varepsilon}}\right)\right]$$

$$+\varepsilon^4\left[-B_{00}^{(4)}\frac{y^2}{2}+B_{11}^{(4)}\cos mx\cos ny+B_{20}^{(4)}\cos 2mx+B_{02}^{(4)}\cos 2ny\right.$$

$$\left.+B_{13}^{(4)}\cos mx\cos 3ny\right]+O(\varepsilon^5) \tag{24}$$

Note that, all of the coefficients in eqns (23) and (24) are related and can be written as the functions of $A_{11}^{(2)}$ but, for the sake of brevity, the detailed expressions are not shown, whereas $\alpha$, $\phi$ and $b$ are given in detail in Appendix B.

Next, substituting eqns (23) and (24) into boundary condition (17c) and into closed condition (16) and into eqn (15), the postbuckling equilibrium paths for the initially heated shells can be written as

$$\lambda_P = \left(1-\frac{T_0}{T_{\rm cr}}\right)\lambda_P^{(0)}-\lambda_P^{(1)}(A_{11}^{(2)}\varepsilon)-\lambda_P^{(2)}(A_{11}^{(2)}\varepsilon)^2+\lambda_P^{(3)}(A_{11}^{(2)}\varepsilon)^3+\lambda_P^{(4)}(A_{11}^{(2)}\varepsilon)^4+\cdots \tag{25}$$

and

$$\delta_x = \delta_P^{(0)}+\delta_P^{(2)}(A_{11}^{(2)}\varepsilon)^2+\delta_P^{(4)}(A_{11}^{(2)}\varepsilon)^4+\cdots \tag{26}$$

in eqns (25) and (26), $(A_{11}^{(2)}\varepsilon)$ is taken as the second perturbation parameter relating to the dimensionless maximum deflection. If the maximum deflection is assumed to be at the point $(x,y) = (0, 0)$, then

$$A_{11}^{(2)}\varepsilon = W_m - \Theta_1 W_m^2 + \cdots \tag{27a}$$

where $W_m$ is the dimensionless form of the maximum deflection of the shell that can be written as

$$W_m = \frac{1}{C_3}\left[\frac{t}{\sqrt[4]{D_{11}^* D_{22}^* A_{11}^* A_{22}^*}}\frac{\bar{W}}{t} + \Theta_2 \lambda_P^{(0)}\right] \tag{27b}$$

All symbols used in eqns (25)–(27) and eqns (28)–(29) below are described in detail in Appendix B.

Similarly, substituting eqns (23) and (24) into boundary condition (17d) and into closed condition (16), the thermal postbuckling equilibrium path for the initially compressed shells can be written as

$$\lambda_T = C_{11}\left[\left(1-\frac{P_x}{P_{\mathrm{cr}}}\right)\lambda_T^{(0)} - \lambda_T^{(1)}(A_{11}^{(2)}\varepsilon) - \lambda_T^{(2)}(A_{11}^{(2)}\varepsilon)^2 + \lambda_T^{(3)}(A_{11}^{(2)}\varepsilon)^3 + \lambda_T^{(4)}(A_{11}^{(2)}\varepsilon)^4 + \cdots\right] \tag{28}$$

in eqn (28) $(A_{11}^{(2)}\varepsilon)$ is also taken as the second perturbation parameter in this case, and we have

$$A_{11}^{(2)}\varepsilon = W_m - \Theta_3 W_m^2 + \cdots \tag{29a}$$

and the dimensionless maximum deflection is written as

$$W_m = \frac{1}{C_3}\left[\frac{t}{\sqrt[4]{D_{11}^* D_{22}^* A_{11}^* A_{22}^*}}\frac{\bar{W}}{t} + \Theta_4 \lambda_T^{(0)}\right] \tag{29b}$$

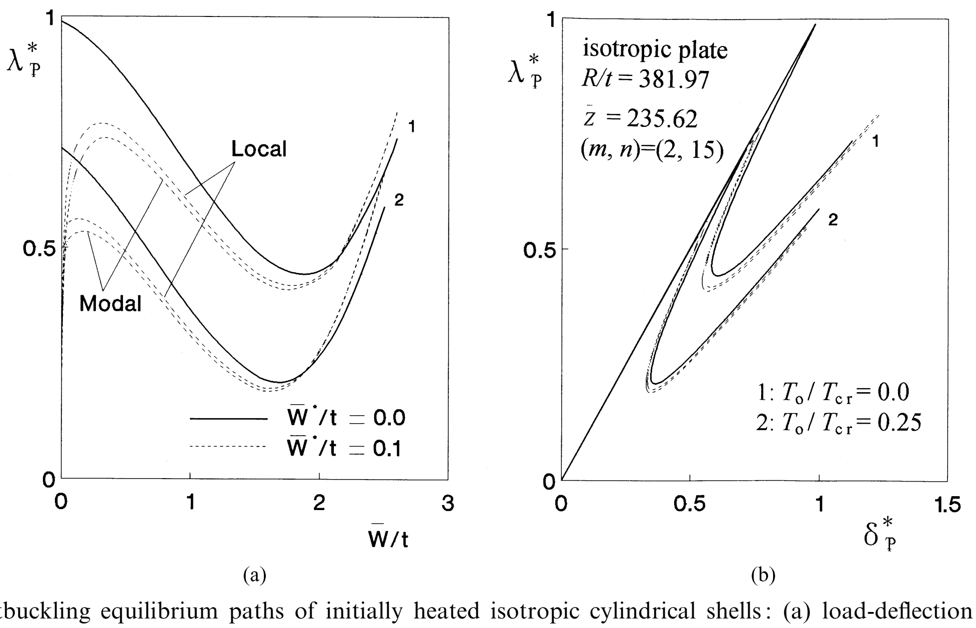
Note that eqns (25) and (28) contain terms $\lambda_P^{(1)}$, $\lambda_P^{(3)}$, $\lambda_T^{(1)}$ and $\lambda_T^{(3)}$ which are not included in the modal imperfection case of Shen (1997c).

Equations (25)–(29) can be used to obtain numerical results for the postbuckling load-deflection (or load-shortening) curves of composite laminated cylindrical shells with local geometric imperfections subjected to combined axial compression and uniform thermal loading, specially for the two cases of compressive postbuckling of initially heated laminated shells; and thermal postbuckling of initially compressed laminated shells. Buckling under pure axial compression and buckling under pure uniform thermal loading follow as two limiting cases. From eqn (11) and Appendix B, equations for the critical value of compressive load $P_{\mathrm{cr}}$ or temperature rise $T_{\mathrm{cr}}$ can easily be found. Due to the perturbation procedure only the same terms of buckling mode in $W^*(x, y, \varepsilon)$ will make contribution to the postbuckling loads, so that $\lambda_P$ (or $\lambda_T$) only depends on $d_{11}$, $d_{20}$, $d_{02}$, $d_{13}$ and $d_{04}$. Because the transverse shear deformation effects are neglected in the present analysis, the shell radius to thickness ratio should be greater than 50 and whose in-plane elastic modulus to shear modulus ratio should be less than 50, i.e. $R/t > 50$ and $E_{11}/G_{12} < 50$. The buckling load of a perfect shell can also be readily obtained numerically, by setting $\mu = 0$ (or $\bar{W}^*/t = 0$), while taking $W_m = 0$ (or $\bar{W}/t = 0$). In all cases, the minimum buckling load is determined by applying eqn (25) or (28) for various values of the buckling mode $(m, n)$, which determine the number of half-waves in the $X$-direction and full waves in the $Y$-direction. Note that because of eqn (23), the prebuckling deformation of the shell is nonlinear, thus the result presented is different from the classical one.

## 4. Numerical results and discussion

Numerical results were obtained to determine the effect of local geometric imperfections on the buckling and postbuckling of composite laminated cylindrical shells under combined axial and thermal loads. A number of examples were solved to illustrate the performance of cross-ply laminated cylindrical shells with or without local or initial buckling modal imperfections, i.e. $W^*(x,y,\varepsilon) = \varepsilon^2 a_m \cos mx \cos ny$, for which the results were obtained numerically in the manner described previously and detailed further in Shen (1997c). Typical results are presented in dimensionless graphical form in which $\lambda_P^*$ and $\delta_P^*$ are used for initially heated shells and $\lambda_T^*$ is used for initially compressed shells. For all of the examples the shells buckled in the asymmetric mode and all plies of equal thickness (note that $t$ remains constant, so that ply thickness $t/N$ decreases as $N$ increases). The shell geometric parameters were $L = 300$ cm, $R = 4L/\pi = 381.97$ cm and total thickness $t = 1$ cm; and the local imperfection parameter were $C_1/L = C_2/R = 0.05$; and the material properties were $E_{11} = 130.3$ GPa, $E_{22} = 9.377$ GPa, $G_{12} = 4.502$ GPa, $\nu_{12} = 0.33$, $\alpha_{11} = 0.139 \times 10^{-6}/°C$ and $\alpha_{22} = 9.0 \times 10^{-6}/°C$. On all figures $\bar{W}^*/t$ and $\bar{W}/t$ mean the dimensionless forms of the maximum values of, respectively, the initial and additional deflection of the shell.

Figures 2–4 show, respectively, the compressive postbuckling load-deflection and load-shortening curves of perfect and imperfect, isotropic and 4-ply $(0/90)_S$ symmetrically and $(0/90)_{2T}$ antisymmetrically cross-ply laminated cylindrical shells for the different values of the initial thermal loading $T_0$ shown. In Figs 2–4, the well-known "snap-through" behavior of shells can be found and the imperfection sensitivity can be predicted. Clearly increasing the initial thermal stress reduces the compressive buckling load substantially and the postbuckling equilibrium path becomes significantly lower. Also the maximum value of $\lambda_P^*$ for the shells with local imperfections is slightly




(a) (b)

Fig. 2. Postbuckling equilibrium paths of initially heated isotropic cylindrical shells: (a) load-deflection; (b) load-shortening.

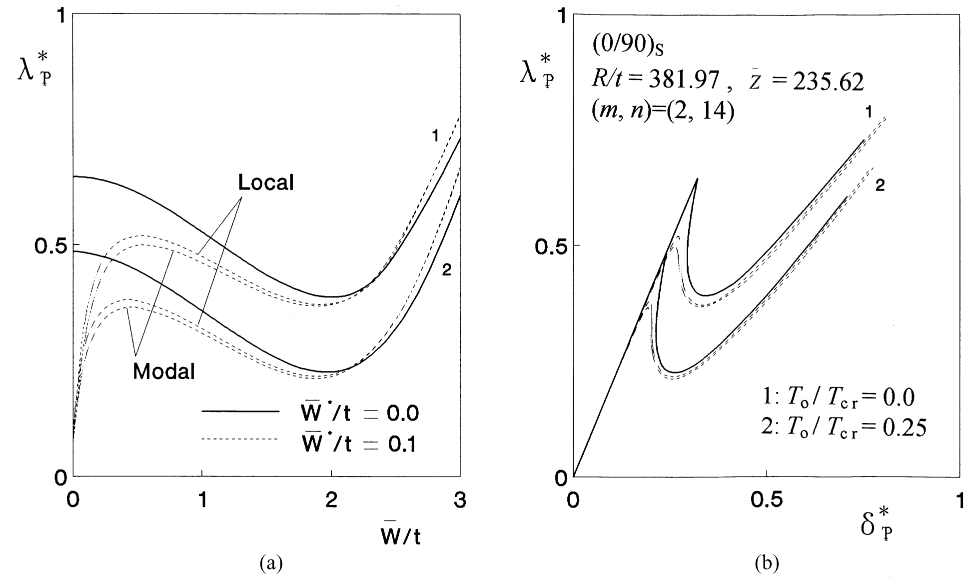



Fig. 3. Postbuckling equilibrium paths of initially heated $(0/90)_S$ laminated cylindrical shells: (a) load-deflection; (b) load-shortening.

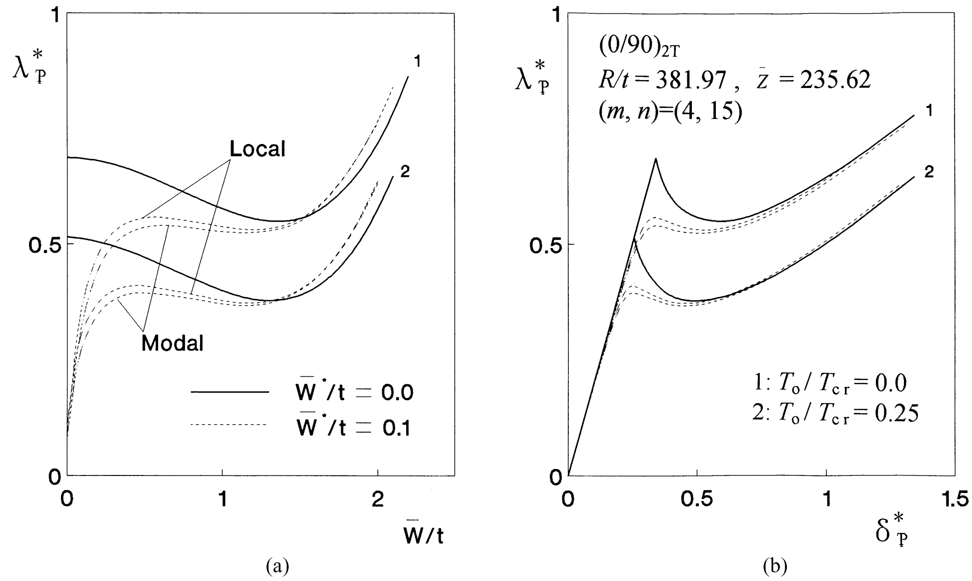



Fig. 4. Postbuckling equilibrium paths of initially heated $(0/90)_{2T}$ laminated cylindrical shells: (a) load-deflection; (b) load-shortening.

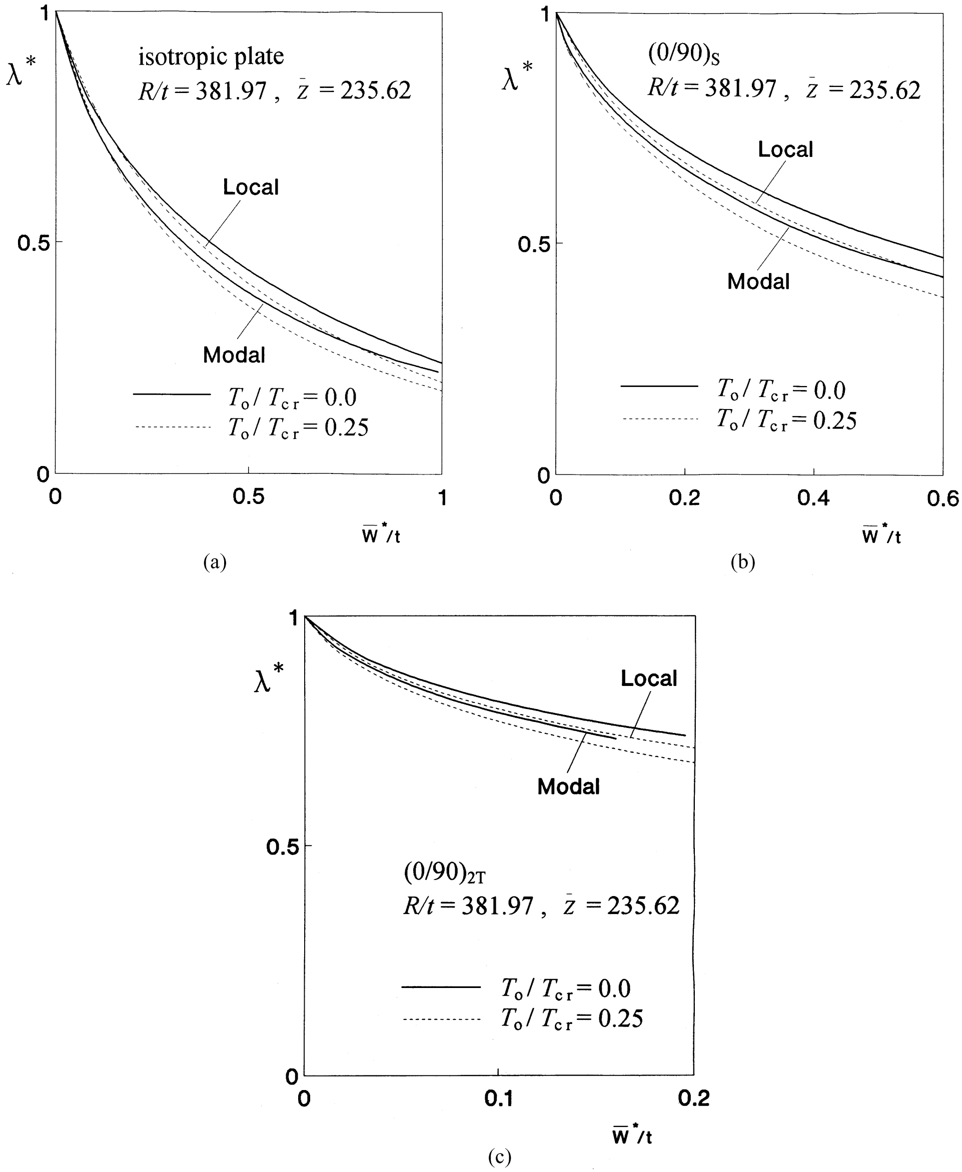



Fig. 5. Comparisons of imperfection sensitivities of initially heated cylindrical shells under axial compression: (a) isotropic; (b) $(0/90)_S$; (c) $(0/90)_{2T}$.

greater than that of the shells with modal imperfections and the discrepancy between these two shape imperfections will become very small when the deflection is sufficiently large.

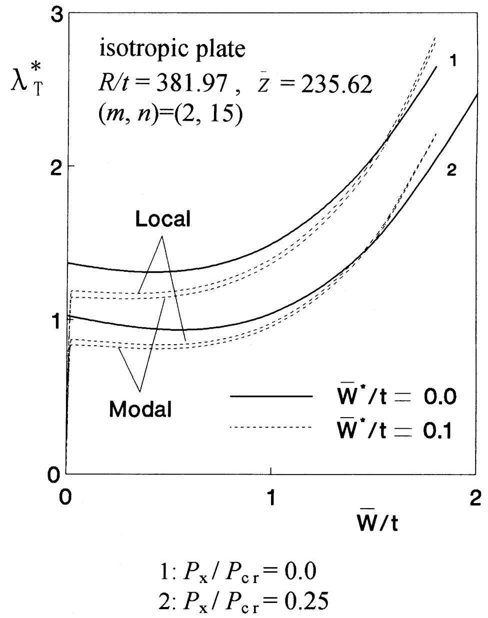
Figure 5 shows curves of imperfection sensitivity for initially heated, isotropic and cross-ply laminated cylindrical shells. $\lambda^*$ is the maximum value of $\lambda_p^*$ as $\bar{W}/t$ varies on curves such as those of Figs 2–4, made dimensionless by dividing by the critical value of $\lambda_P^*$ for the perfect shell, i.e. by dividing the $\lambda_P^*$ given by $\bar{W}/t = \bar{W}^*/t = 0.0$. These results show that the imperfection sensitivity of the isotropic shell is the largest one, and the imperfection sensitivity of the $(0/90)_S$ shell is greater than that of $(0/90)_{2T}$. The imperfection sensitivity of shells with local imperfections is weaker than that of shells with modal imperfections. Also the imperfection sensitivity of an initial heated shell is slightly greater than that of the shell without any initial thermal stress.

Figures 6–8 are the thermal postbuckling results for initially compressed shells analogous to the compressive postbuckling results of Figs 2–4, but without load-shortening curves.

Figure 6 shows the thermal postbuckling load-deflection curves of isotropic cylindrical shells with or without local or modal imperfections. Then Fig. 7 shows that the perfect $(0/90)_S$ laminated cylindrical shells has a weak "snap-through" thermal postbuckling response. In contrast, for an imperfect shell with local or modal imperfections there is no maximum value of $\lambda_T^*$ when $\bar{W}^*/t > 0.08$ (or 0.06), as a result the shell structure becomes imperfection-insensitive.

Figure 8 shows that if the temperature exceeds a critical buckling level, the thermal postbuckling load-deflection curves of the initially compressed $(0/90)_{2T}$ laminated shell go upward dramatically, and the shell structure also becomes imperfection-insensitive.

Figure 9 shows curves of imperfection sensitivity for initially compressed $(0/90)_S$ laminated cylindrical shells. Now $\lambda^*$ is the maximum value of $\lambda_T^*$ made dimensionless by dividing by the critical value of $\lambda_T^*$ for the perfect shell. Note that in all of these examples the imperfection sensitivity of shells under thermal load is weaker than that of shells under compressive load (compare Figs 5(b) and 9), and the results shown in Fig. 9 were only for the $(0/90)_S$ laminated cylindrical shell with very small initial geometric imperfections.




1: $P_x / P_{cr} = 0.0$
2: $P_x / P_{cr} = 0.25$

Fig. 6. Thermal postbuckling load-deflection curves of initially compressed isotropic cylindrical shells.

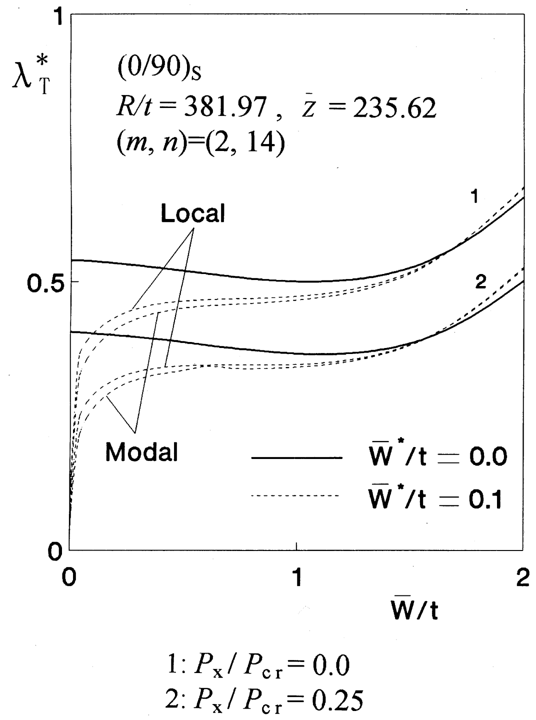



Fig. 7. Thermal postbuckling load-deflection curves of initially compressed $(0/90)_S$ laminated cylindrical shells.

## 5. Conclusions

In order to assess the effect of local geometric imperfections on the buckling and postbuckling of composite laminated cylindrical shells subjected to combined axial compression and a uniform temperature rise, a postbuckling analysis is developed according to a boundary layer theory and a fully nonlinear postbuckling load-deflection or load-shortening curve is presented. The two cases of compressive postbuckling of initially heated shells and of thermal postbuckling of initially compressed shells have been considered. The numerical examples relate to the performances of cross-ply laminated cylindrical shells with or without local geometric imperfections, from which results for isotropic cylindrical shells follow as a limiting case. Like the shell with initial buckling modal imperfections, the numerical results show that the postbuckling behavior of initially compressed, laminated cylindrical shells under thermal load is different from that of mechanically loaded shells with and without initial thermal stress. In many cases the cylindrical shell has a stable thermal postbuckling equilibrium path, and the shell structure becomes imperfection-insensitive. In addition, for a given value of $\bar{W}^*/t$, local imperfections have an effect which is somewhat less severe than a modal imperfection, and the imperfection sensitivity of shells with local imperfections is weaker than that of shells with modal imperfections.

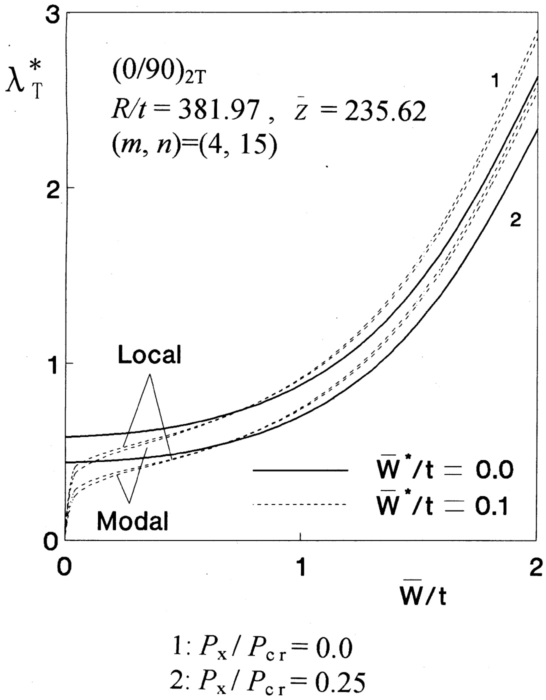



Fig. 8. Thermal postbuckling load-deflection curves of initially compressed $(0/90)_{2T}$ laminated cylindrical shells.

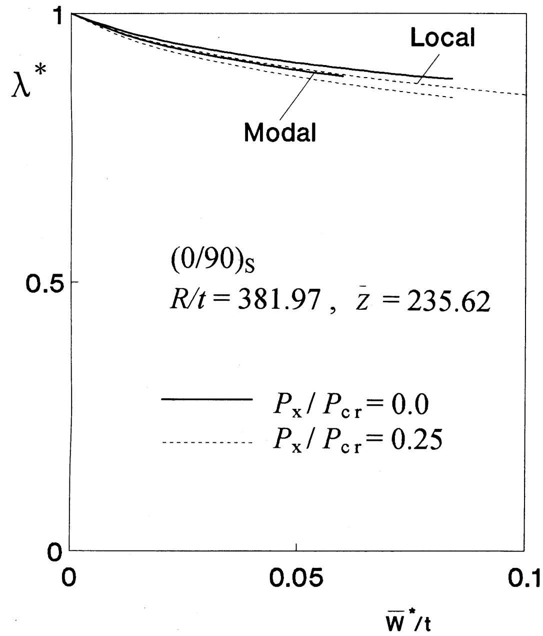



Fig. 9. Comparisons of imperfection sensitivities of initially compressed $(0/90)_S$ laminated cylindrical shells under uniform temperature loading.

**Appendix A**

The membrane, coupling and flexural rigidities beneath eqn (3) are defined as

$$\mathbf{A}=\begin{bmatrix}A_{11} & A_{12} & 0\\ A_{12} & A_{22} & 0\\ 0 & 0 & A_{66}\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}B_{11} & B_{12} & 0\\ B_{12} & B_{22} & 0\\ 0 & 0 & B_{66}\end{bmatrix},\quad \mathbf{D}=\begin{bmatrix}D_{11} & D_{12} & 0\\ D_{12} & D_{22} & 0\\ 0 & 0 & D_{66}\end{bmatrix} \tag{A1}$$

and $A_{ij}$, $B_{ij}$ and $D_{ij}$ are defined by

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}\int_{t_{k-1}}^{t_k} (\bar{Q}_{ij})_k(1, Z, Z^2)\,\mathrm{d}Z \quad (i,j=1,2,6) \tag{A2}$$

where $\bar{Q}_{ij}$ are the transformed elastic constants, defined by

$$\begin{bmatrix}\bar{Q}_{11}\\ \bar{Q}_{12}\\ \bar{Q}_{22}\\ \bar{Q}_{16}\\ \bar{Q}_{26}\\ \bar{Q}_{66}\end{bmatrix}=\begin{bmatrix}c^4 & 2c^2s^2 & s^4 & 4c^2s^2\\ c^2s^2 & c^4+s^4 & c^2s^2 & -4c^2s^2\\ s^4 & 2c^2s^2 & c^4 & 4c^2s^2\\ c^3s & cs^3-c^3s & -cs^3 & -2cs(c^2-s^2)\\ cs^3 & c^3s-cs^3 & -c^3s & 2cs(c^2-s^2)\\ c^2s^2 & -2c^2s^2 & c^2s^2 & (c^2-s^2)^2\end{bmatrix}\begin{bmatrix}Q_{11}\\ Q_{12}\\ Q_{22}\\ Q_{66}\end{bmatrix} \tag{A3}$$

where

$$Q_{11}=\frac{E_{11}}{(1-v_{12}v_{21})},\quad Q_{22}=\frac{E_{22}}{(1-v_{12}v_{21})},\quad Q_{12}=\frac{v_{21}E_{11}}{(1-v_{12}v_{21})},\quad Q_{66}=G_{12} \tag{A4}$$

and

$$c=\cos\theta,\quad s=\sin\theta \tag{A5}$$

where $\theta$ = lamination angle with respect to the shell $X$-axis.

$$\begin{bmatrix}A_x\\ A_y\\ A_{xy}\end{bmatrix}=-\begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26}\\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}\end{bmatrix}\begin{bmatrix}c^2 & s^2\\ s^2 & c^2\\ 2cs & -2cs\end{bmatrix}\begin{bmatrix}\alpha_{11}\\ \alpha_{22}\end{bmatrix} \tag{A6}$$

where $\alpha_{11}$ and $\alpha_{22}$ are thermal expansion coefficients for a single ply.

**Appendix B**

In eqns (25)–(27)

$$\Theta_1=\frac{1}{C_3}\left[\left(\frac{\gamma_{24}^2}{\gamma_{14}\gamma_{24}+\gamma_{34}^2}\right)\frac{m^4\mu_{11}}{16n^2\beta^2 g_2}\varepsilon^{-1}+\frac{1}{32}\left(\frac{\gamma_{24}^2}{\gamma_{14}\gamma_{24}+\gamma_{34}^2}\right)\frac{m^2}{\gamma_{24}n^2\beta^2}\left(\frac{\gamma_{34}}{\gamma_{24}}\mu_{12}-\frac{2\gamma_{24}g_3}{g_2}\right)+\frac{2\gamma_5}{\gamma_{24}}\lambda_P^{(2)}\right]$$

$$\Theta_2 = 2\left[\frac{\gamma_5}{\gamma_{24}}\left(1-\frac{T_0}{T_{cr}}\right)+\frac{\gamma_{24}^2-\gamma_5^2}{\gamma_{24}}\frac{\gamma_{T2}}{(\gamma_{24}^2\gamma_{T1}-\gamma_5\gamma_{T2})+\frac{4}{\pi}\frac{\alpha}{b}\gamma_5(\gamma_{T2}-\gamma_5\gamma_{T1})\varepsilon^{1/2}}\frac{T_0}{T_{cr}}\right]$$

$$C_3 = 1-\frac{g_3}{m^2}\varepsilon-(\mu d_{20})+\frac{2\gamma_5}{\gamma_{24}}\lambda_P^{(1)}$$

$$\lambda_P^{(0)} = \frac{1}{2}\left[\frac{\gamma_{24}m^2}{g_2\mu_{11}}\varepsilon^{-1}+\frac{\gamma_{24}g_3}{g_2}\mu_{101}+\frac{1}{m^2\mu_{11}}\left(\frac{g_1}{\gamma_{14}}+\frac{\gamma_{24}g_3^2}{g_2}\mu_{201}\right)\varepsilon\right.$$
$$\left.-\mu_{301}\frac{g_3}{m^4}\left(1+\frac{g_3}{m^2\mu_{11}}\varepsilon\right)\left(\frac{g_1}{\gamma_{14}}+\frac{\gamma_{24}g_3^2}{g_2}\mu_{302}\right)\varepsilon^2\right]$$

$$\lambda_P^{(1)} = \frac{1}{2}\left[4\frac{\gamma_{24}m^2n^2\beta^2}{g_2\mu_{11}}(\mu d_{02})+\left(2\frac{\gamma_{24}n^2\beta^2g_3}{g_2}\frac{\mu_{21}}{\mu_{11}}(\mu d_{02})\right)\varepsilon\right.$$
$$\left.+\left(8\frac{m^2n^2\beta^2}{\gamma_{14}}\frac{g_2\mu_{11}+4m^4\mu_{21}}{g_2\mu_{11}-4m^4}(\mu d_{20})\right)\varepsilon^2\right]$$

$$\lambda_P^{(2)} = \frac{1}{8}\left[\left(\frac{\gamma_{24}^2}{\gamma_{14}\gamma_{24}+\gamma_{34}^2}\right)\frac{\gamma_{24}m^6\mu_{213}}{2g_2^2}\varepsilon^{-1}+\left(\frac{\gamma_{24}^2}{\gamma_{14}\gamma_{24}+\gamma_{34}^2}\right)\frac{m^4}{2g_2}\left(\frac{\gamma_{34}}{\gamma_{24}}\mu_{321}\right.\right.$$
$$\left.-\frac{\gamma_{24}g_3}{g_2}\mu_{322}\right)-\frac{1}{4}\left(\frac{\gamma_{14}}{\gamma_{14}\gamma_{24}+\gamma_{34}^2}\right)m^2\mu_{222}\varepsilon+\left(\frac{\gamma_{24}m^2n^4\beta^4}{g_2}\frac{g_2\mu_{421}+8m^4\mu_{422}}{g_2\mu_{11}-4m^4}\right)\varepsilon$$
$$\left.-\left(\frac{\gamma_{24}^2}{\gamma_{14}\gamma_{24}+\gamma_{34}^2}\right)\frac{m^2g_3}{4g_2}\left(\frac{\gamma_{34}}{\gamma_{24}}\mu_{423}+\frac{2\gamma_{24}g_3}{g_2}\mu_{424}\right)\varepsilon\right]$$

$$\lambda_P^{(3)} = \frac{1}{8}\left[\left(\frac{\gamma_{24}^2}{\gamma_{14}\gamma_{24}+\gamma_{34}^2}\right)\frac{\gamma_{24}m^6n^2\beta^2}{g_2^2}\frac{g_{13}\mu_{431}+g_2\mu_{432}}{g_{13}-g_2\mu_{11}}(\mu d_{02})\right.$$
$$\left.+4\frac{\gamma_{24}m^6n^2\beta^2}{g_2^2}\frac{g_{13}+g_2\mu_{31}}{g_{13}-g_2\mu_{11}}\mu_{11}(\mu d_{04})\right]$$

$$\lambda_P^{(4)} = \frac{1}{128}\left(\frac{\gamma_{24}^2}{\gamma_{14}\gamma_{24}+\gamma_{34}^2}\right)^2\frac{\gamma_{24}m^{10}\mu_{11}}{g_2^3}\frac{g_{13}\mu_{441}+g_2\mu_{442}}{g_{13}-g_2\mu_{11}}\varepsilon^{-1}$$

$$\delta_P^{(0)} = \left[\gamma_{24}-\frac{4}{\pi}\frac{\alpha}{b}\frac{\gamma_5^2}{\gamma_{24}}\varepsilon^{1/2}\right]\lambda_P+\left[\frac{\gamma_5^2}{2\pi\gamma_{24}^2}\frac{b}{\alpha}\varepsilon^{1/2}\right]\lambda_P^2$$

$$\delta_P^{(2)} = \frac{1}{16}\left[m^2\left(\mu_{12}-8(\mu d_{20})^2+\frac{1}{\alpha\pi}\varepsilon^{1/2}\right)\varepsilon-2g_3\varepsilon^2+\frac{g_3^2}{m^2}\varepsilon^3\right]$$

$$\delta_P^{(4)} = \frac{1}{128}\left[\frac{1}{32\pi\alpha}\left(\frac{\gamma_{24}^2}{\gamma_{14}\gamma_{24}+\gamma_{34}^2}\right)^2 \frac{m^8\mu_{11}^2}{n^4\beta^4 g_2^2}\varepsilon^{-3/2} + m^2n^4\beta^4\mu_{11}^2\left(\frac{g_2\mu_{12}+8m^4\mu_{11}}{g_2\mu_{11}-4m^4}\right)^2\varepsilon^3\right] \tag{B1}$$

and in eqns (28) and (29)

$$C_{11} = \frac{\gamma_{24}^2 - \dfrac{4}{\pi}\dfrac{\alpha}{b}\gamma_5^2\varepsilon^{1/2}}{(\gamma_{24}^2\gamma_{T1}-\gamma_5\gamma_{T2}) + \dfrac{4}{\pi}\dfrac{\alpha}{b}\gamma_5(\gamma_{T2}-\gamma_5\gamma_{T1})\varepsilon^{1/2}}$$

$$\Theta_3 = \frac{1}{C_3}\left\{\left(\frac{\gamma_{24}^2}{\gamma_{14}\gamma_{24}+\gamma_{34}^2}\right)\frac{m^4\mu_{11}}{16n^2\beta^2 g_2}\varepsilon^{-1} + \frac{1}{32}\left(\frac{\gamma_{24}^2}{\gamma_{14}\gamma_{24}+\gamma_{34}^2}\right)\frac{m^2}{\gamma_{24}n^2\beta^2}\left(\frac{\gamma_{34}}{\gamma_{24}}\mu_{12} - \frac{2\gamma_{24}g_3}{g_2}\right)\right.$$

$$+\frac{1}{8}\frac{\gamma_5}{\gamma_{24}^2 - \dfrac{4}{\pi}\dfrac{\alpha}{b}\gamma_5^2\varepsilon^{1/2}}\left[m^2\left(\mu_{12} - 8(\mu d_{20})^2 + \frac{1}{\alpha\pi}\varepsilon^{1/2}\right)\varepsilon - 2g_3\varepsilon^2 + \frac{g_3^2}{m^2}\varepsilon^3\right]$$

$$\left.+\frac{\gamma_{24}^2-\gamma_5^2}{\gamma_{24}}\frac{\gamma_{T2}}{(\gamma_{24}^2\gamma_{T1}-\gamma_5\gamma_{T2}) + \dfrac{4}{\pi}\dfrac{\alpha}{b}\gamma_5(\gamma_{T2}-\gamma_5\gamma_{T1})\varepsilon^{1/2}}\lambda_T^{(2)}\right\}$$

$$\Theta_4 = \frac{\gamma_5}{\gamma_{24}}\frac{P_x}{P_{cr}} + \frac{\gamma_{24}^2-\gamma_5^2}{\gamma_{24}}\frac{\gamma_{T2}}{(\gamma_{24}^2\gamma_{T1}-\gamma_5\gamma_{T2}) + \dfrac{4}{\pi}\dfrac{\alpha}{b}\gamma_5(\gamma_{T2}-\gamma_5\gamma_{T1})\varepsilon^{1/2}}\left(1 - \frac{P_x}{P_{cr}}\right)$$

$$C_3 = 1 - \frac{g_3}{m^2}\varepsilon - (\mu d_{20}) + \frac{\gamma_{24}^2-\gamma_5^2}{\gamma_{24}}\frac{\gamma_{T2}}{(\gamma_{24}^2\gamma_{T1}-\gamma_5\gamma_{T2}) + \dfrac{4}{\pi}\dfrac{\alpha}{b}\gamma_5(\gamma_{T2}-\gamma_5\gamma_{T1})\varepsilon^{1/2}}\lambda_T^{(1)}$$

$$\lambda_T^{(0)} = 2\lambda_P^{(0)}$$

$$\lambda_T^{(1)} = 2\lambda_P^{(1)}$$

$$\lambda_T^{(2)} = 2\lambda_P^{(2)} - \frac{1}{8}\frac{\gamma_{24}}{\gamma_{24}^2 - \dfrac{4}{\pi}\dfrac{\alpha}{b}\gamma_5^2\varepsilon^{1/2}}\left[m^2\left(\mu_{12} - 8(\mu d_{20})^2 + \frac{1}{\alpha\pi}\varepsilon^{1/2}\right)\varepsilon - 2g_3\varepsilon^2 + \frac{g_3^2}{m^2}\varepsilon^3\right]$$

$$\lambda_T^{(3)} = 2\lambda_P^{(3)}$$

$$\lambda_T^{(4)} = 2\lambda_P^{(4)} + \frac{1}{64}\frac{\gamma_{24}}{\gamma_{24}^2 - \dfrac{4}{\pi}\dfrac{\alpha}{b}\gamma_5^2\varepsilon^{1/2}}\left[\frac{1}{32\pi\alpha}\left(\frac{\gamma_{24}^2}{\gamma_{14}\gamma_{24}+\gamma_{34}^2}\right)^2\frac{m^8\mu_{11}^2}{n^4\beta^4 g_2^2}\varepsilon^{-3/2}\right.$$

$$\left.+m^2n^4\beta^4\mu_{11}^2\left(\frac{g_2\mu_{12}+8m^4\mu_{11}}{g_2\mu_{11}-4m^4}\right)^2\varepsilon^3\right] \tag{B2}$$

in the preceding equations

$$g_1 = m^4 + 2\gamma_{12} m^2 n^2 \beta^2 + \gamma_{14}^2 n^4 \beta^4$$
$$g_2 = m^4 + 2\gamma_{22} m^2 n^2 \beta^2 + \gamma_{24}^2 n^4 \beta^4$$
$$g_3 = \gamma_{30} m^4 + \gamma_{32} m^2 n^2 \beta^2 + \gamma_{34} n^4 \beta^4$$
$$g_{13} = m^4 + 18\gamma_{22} m^2 n^2 \beta^2 + 81\gamma_{24}^2 n^4 \beta^4$$
$$\alpha = [\tfrac{1}{2}(b-c)]^{1/2}, \quad \phi = [\tfrac{1}{2}(b+c)]^{1/2}$$
$$b = \left[\frac{\gamma_{14}\gamma_{24}}{1+\gamma_{14}\gamma_{24}\gamma_{30}^2}\right]^{1/2}, \quad c = -\frac{\gamma_{14}\gamma_{24}\gamma_{30}}{1+\gamma_{14}\gamma_{24}\gamma_{30}^2} \tag{B3}$$

and

$$\mu_{11} = 1+\mu d_{11}, \quad \mu_{21} = 2+\mu d_{11}, \quad \mu_{31} = 3+\mu d_{11}, \quad \mu_{41} = 4+\mu d_{11}, \quad \mu_{12} = 1+2\mu d_{11},$$
$$\mu_{113} = 1+\mu d_{11}+\mu d_{13}, \quad \mu_{213} = 1+\mu_{113}, \quad \mu_{201} = \mu_{101}-1,$$
$$\mu_{101} = \frac{\mu_{21}}{\mu_{11}^2}, \quad \mu_{301} = \frac{\mu d_{11}}{\mu_{11}^2}, \quad \mu_{302} = \frac{\mu_{21}^2}{\mu_{11}^2}, \quad \mu_{222} = \frac{\mu_{12}\mu_{113}}{\mu_{11}},$$
$$\mu_{321} = \frac{\mu_{21}}{\mu_{11}} + \mu_{11}, \quad \mu_{322} = \frac{2-\mu_{113}}{\mu_{11}} - \mu_{113},$$
$$\mu_{421} = \mu_{12}\mu_{31} + 2\mu_{11}^2, \quad \mu_{422} = \mu_{11}\mu_{21},$$
$$\mu_{423} = \frac{\mu_{21}}{\mu_{11}}\mu_{12} + 2\frac{\mu_{113}}{\mu_{11}} + 2\mu_{113} - \frac{\mu_{12}\mu_{41}}{\mu_{11}^2} + 2, \quad \mu_{424} = \frac{5-\mu_{11}}{\mu_{11}^2} - \frac{\mu_{113}^2}{\mu_{11}} - 1$$
$$\mu_{431} = 2\mu_{11} + \frac{(1+\mu_{113})(2\mu_{11}+\mu_{113})}{\mu_{11}}, \quad \mu_{432} = \frac{\mu_{21}^3}{\mu_{11}} + \frac{\mu_{113}}{\mu_{11}} - \mu_{113}(\mu_{113}+\mu d_{11})$$
$$\mu_{441} = \mu_{21} + 2\mu_{113} + \frac{\mu_{113}^2}{\mu_{11}} + \frac{\mu_{113}^3}{\mu_{11}}, \quad \mu_{442} = \mu_{21}\mu_{31} + 2\mu_{113} - \mu_{113}^2 - \mu_{113}^3$$
$$d_{11} = \frac{8}{\pi^2}\frac{\gamma_2}{(\gamma_1^2+m^2)(\gamma_2^2+n^2)}\left[\gamma_1 + \left(m\sin\frac{m\pi}{2} - \gamma_1\cos\frac{m\pi}{2}\right)\exp(-\gamma_1\pi/2)\right]$$
$$\times[1-(-1)^n\exp(-\gamma^2\pi)]$$
$$d_{13} = \frac{8}{\pi^2}\frac{\gamma_2}{(\gamma_1^2+m^2)(\gamma_2^2+9n^2)}\left[\gamma_1 + \left(m\sin\frac{m\pi}{2} - \gamma_1\cos\frac{m\pi}{2}\right)\exp(-\gamma_1\pi/2)\right]$$
$$\times[1-(-1)^n\exp(-\gamma_2\pi)]$$
$$d_{20} = \frac{4}{\pi^2\gamma_2}\frac{\gamma_1}{(\gamma_1^2+4m^2)}[1-(-1)^m\exp(-\gamma_1\pi/2)][1-\exp(-\gamma_2\pi)]$$
$$d_{02} = \frac{4}{\pi^2\gamma_1}\frac{\gamma_2}{(\gamma_2^2+4n^2)}[1-\exp(-\gamma_1\pi/2)][1-\exp(-\gamma_2\pi)]$$

$$d_{04} = \frac{4}{\pi^2 \gamma_1} \frac{\gamma_2}{(\gamma_2^2 + 16n^2)} [1 - \exp(-\gamma_1 \pi/2)][1 - \exp(-\gamma_2 \pi)] \tag{B4}$$

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